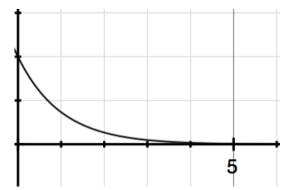
# Bateman Equations

## Parent Isotope

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \implies N_1 = \left[ N_1|_{t=0} \right] e^{-\lambda_1 t} \tag{1}$$

After 5 lifetimes the parent is nearly gone.

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## Daughter Isotope

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \implies \frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 \left[ N_1|_{t=0} \right] e^{-\lambda_1 t} \tag{2}$$

$$N_2 = \frac{\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)}{\lambda_2 - \lambda_1} \left\{ \lambda_1 \left[ N_1|_{t=0} \right] \right\} + \left[ N_2|_{t=0} \right] e^{-\lambda_2 t} \tag{3}$$

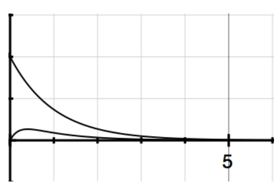
$$\frac{dN_2}{dt} = -\frac{\lambda_1^2}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] e^{-\lambda_2 t} - \lambda_2 \left[ N_2|_{t=0} \right] e^{-\lambda_2 t}$$

$$\lambda_2 N_2 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] e^{-\lambda_1 t} - \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] e^{-\lambda_2 t} + \lambda_2 \left[ N_2|_{t=0} \right] e^{-\lambda_2 t}$$

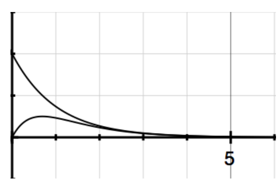
$$N_2 = \left\{ \frac{\lambda_1}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] \right\} e^{-\lambda_1 t} + \left\{ \left[ N_2|_{t=0} \right] - \frac{\lambda_1}{\lambda_2 - \lambda_1} \left[ N_1|_{t=0} \right] \right\} e^{-\lambda_2 t} \tag{4}$$

### Transient Equilibrium

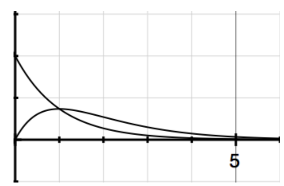
$$\text{Set} \ , \ \frac{dN_2}{dt} = 0 \ , \ \text{with} \ , \ \left[ N_1|_{t=0} \right] = 0 \implies t_{\max} = \frac{1}{\lambda_2 - \lambda_1} \ln \frac{\lambda_2}{\lambda_1} \tag{5}$$



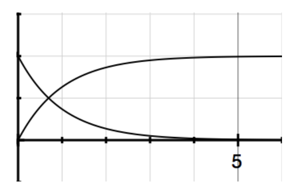

$\lambda_2 \approx 5 \times \lambda_1$




$\lambda_2 \approx 2 \times \lambda_1$




$\lambda_2 \approx 1 \times \lambda_1$




$\lambda_2 \approx 0 \times \lambda_1$

### Secular Equilibrium

$$\lambda_2 \gg \lambda_1 \implies N_2 \approx \frac{\lambda_1}{\lambda_2} N_1 \ , \text{from the earliest times.} \tag{6}$$

### For $t$ Small

$$N_2 \approx \left[ N_1|_{t=0} \right] \lambda_1 t + \left[ N_2|_{t=0} \right] (1 - \lambda_2 t) \iff \frac{dN_2}{dt} = \lambda_1 \left[ N_1|_{t=0} \right] - \lambda_2 \left[ N_2|_{t=0} \right] \tag{7}$$

## GrandDaughter Isotope

$$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3 \implies \frac{dN_3}{dt} + \lambda_3 N_3 = \lambda_2 N_2 \tag{8}$$

$$\frac{dN_3}{dt} + \lambda_3 N_3 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ N_1|_0 \right] \left( e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) + \lambda_2 \left[ N_2|_0 \right] e^{-\lambda_2 t} \tag{9}$$

$$\mathbb{A} \equiv \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ N_1|_0 \right] \text{ , and , } \mathbb{B} \equiv \lambda_2 \left[ N_2|_0 \right] - \mathbb{A} \tag{10}$$

$$\frac{dN_3}{dt} + \lambda_3 N_3 = \mathbb{A} e^{-\lambda_1 t} + \mathbb{B} e^{-\lambda_2 t} \tag{11}$$

$$N_3 = \frac{\left( e^{-\lambda_1 t} - e^{-\lambda_3 t} \right)}{\lambda_3 - \lambda_1} \mathbb{A} + \frac{\left( e^{-\lambda_2 t} - e^{-\lambda_3 t} \right)}{\lambda_3 - \lambda_2} \mathbb{B} + \left[ N_3|_0 \right] e^{-\lambda_3 t} \tag{12}$$

$$\frac{dN_3}{dt} = -\frac{\lambda_1}{\lambda_3 - \lambda_1} \mathbb{A} e^{-\lambda_1 t} + \frac{\lambda_3}{\lambda_3 - \lambda_1} \mathbb{A} e^{-\lambda_3 t} - \frac{\lambda_2}{\lambda_3 - \lambda_2} \mathbb{B} e^{-\lambda_2 t} + \frac{\lambda_3}{\lambda_3 - \lambda_2} \mathbb{B} e^{-\lambda_2 t} - \lambda_3 \left[ N_3|_0 \right] e^{-\lambda_3 t}$$

$$\lambda_3 N_3 = \frac{\lambda_3}{\lambda_3 - \lambda_1} \mathbb{A} e^{-\lambda_1 t} - \frac{\lambda_3}{\lambda_3 - \lambda_1} \mathbb{A} e^{-\lambda_3 t} + \frac{\lambda_3}{\lambda_3 - \lambda_2} \mathbb{B} e^{-\lambda_2 t} - \frac{\lambda_3}{\lambda_3 - \lambda_2} \mathbb{B} e^{-\lambda_2 t} + \lambda_3 \left[ N_3|_0 \right] e^{-\lambda_3 t}$$

$$N_3 = \left\{ \frac{\lambda_1}{\lambda_2 - \lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_1} \left[ N_1|_0 \right] \right\} e^{-\lambda_1 t} + \left\{ \frac{\lambda_2}{\lambda_3 - \lambda_2} \left[ N_2|_0 \right] - \frac{\lambda_1}{\lambda_2 - \lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_2} \left[ N_1|_0 \right] \right\} e^{-\lambda_2 t} + \left\{ \left[ N_3|_0 \right] - \frac{\lambda_2}{\lambda_3 - \lambda_2} \left[ N_2|_0 \right] + \frac{\lambda_1}{\lambda_3 - \lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_2} \left[ N_1|_0 \right] \right\} e^{-\lambda_3 t} \tag{13}$$

$$\frac{dN_1}{dt} + \lambda_1 N_1 = 0 \tag{14}$$

$$N_1 = [N_1|_0]\, e^{-\lambda_1 t} \tag{15}$$

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$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 [N_1|_0]\, e^{-\lambda_1 t} \tag{16}$$

$$N_2 = \left\{ \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_1 t} + \left\{ [N_2|_0] - \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_2 t} \tag{17}$$

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$$\frac{dN_3}{dt} + \lambda_3 N_3 = \lambda_2 [N_2|_0]\, e^{-\lambda_2 t} \tag{18}$$

$$\begin{aligned} N_3 = &\left\{ \frac{\lambda_2}{\lambda_3 - \lambda_1} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_1 t} + \left\{ \frac{\lambda_2}{\lambda_3 - \lambda_2} [N_2|_0] - \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_2 t} \\ &+ \left\{ [N_3|_0] - \frac{\lambda_2}{\lambda_3 - \lambda_2} [N_2|_0] + \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_3 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_3 t} \end{aligned} \tag{19}$$

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$$\frac{dN_4}{dt} + \lambda_4 N_4 = \lambda_3 [N_3|_0]\, e^{-\lambda_3 t} \tag{20}$$

$$\begin{aligned} N_4 = &\left\{ \frac{\lambda_3}{\lambda_4 - \lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_1} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_1 t} + \left\{ \frac{\lambda_3}{\lambda_4 - \lambda_2} \frac{\lambda_2}{\lambda_3 - \lambda_2} [N_2|_0] - \frac{\lambda_3}{\lambda_4 - \lambda_2} \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_2 t} \\ &+ \left\{ \frac{\lambda_3}{\lambda_4 - \lambda_3} [N_3|_0] - \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_3 - \lambda_2} [N_2|_0] + \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_3 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_3 t} \\ &+ \left\{ [N_4|_0] - \frac{\lambda_3}{\lambda_4 - \lambda_3} [N_3|_0] + \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_4 - \lambda_2} [N_2|_0] - \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_4 - \lambda_2} \frac{\lambda_1}{\lambda_4 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_4 t} \end{aligned} \tag{21}$$

At the boundary, $N_4(t=0) = [N_4|_0] + (\text{stuff})_3 \times [N_3|_0] + (\text{stuff})_2 \times [N_2|_0] + (\text{stuff})_1 \times [N_1|_0]$

$$(\text{stuff})_3 \times [N_3|_0] = \left\{ \frac{\lambda_3}{\lambda_4 - \lambda_3} - \frac{\lambda_3}{\lambda_4 - \lambda_3} \right\} [N_3|_0] \equiv 0 \tag{22}$$

$$(\text{stuff})_2 \times [N_2|_0] = \left\{ \frac{\lambda_3}{\lambda_4 - \lambda_2} \frac{\lambda_2}{\lambda_3 - \lambda_2} - \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_3 - \lambda_2} + \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_4 - \lambda_2} \right\} [N_2|_0] \equiv 0 \tag{23}$$

$$\begin{aligned}
(\text{stuff})_1 \times [N_1|_0] = \left\{ \frac{\lambda_3}{\lambda_4 - \lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_1} \frac{\lambda_1}{\lambda_2 - \lambda_1} - \frac{\lambda_3}{\lambda_4 - \lambda_2} \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_2 - \lambda_1} + \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_3 - \lambda_1} - \frac{\lambda_3}{\lambda_4 - \lambda_3} \frac{\lambda_2}{\lambda_4 - \lambda_2} \frac{\lambda_1}{\lambda_4 - \lambda_1} \right\} [N_1|_0] &= \\
\left( \frac{\lambda_3 \lambda_2 \lambda_1}{(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} \right) [N_1|_0] \times & \\
\{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3) - (\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3) + (\lambda_4 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)\} &= \\
\left( \frac{\lambda_3 \lambda_2 \lambda_1}{(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} \right) [N_1|_0] \times \{ \; (\lambda_3 - \lambda_2)\lambda_4^2 + (\lambda_2 - \lambda_4)\lambda_3^2 + (\lambda_4 - \lambda_3)\lambda_2^2 & \\
-(\lambda_3 - \lambda_1)\lambda_4^2 - (\lambda_1 - \lambda_4)\lambda_3^2 - (\lambda_4 - \lambda_3)\lambda_1^2 & \\
+(\lambda_2 - \lambda_1)\lambda_4^2 + (\lambda_1 - \lambda_4)\lambda_2^2 + (\lambda_4 - \lambda_2)\lambda_1^2 & \\
-(\lambda_2 - \lambda_1)\lambda_3^2 - (\lambda_1 - \lambda_3)\lambda_2^2 - (\lambda_3 - \lambda_2)\lambda_1^2 \; \} &\equiv 0
\end{aligned} \tag{24}$$

## $\lambda_i = 0$ for $i \geq 2$

$$\frac{dN_1}{dt} + \lambda_1 N_1 = 0 \tag{25}$$

$$N_1 = [N_1|_0] \, e^{-\lambda_1 t} \tag{26}$$

$$\frac{dN_2}{dt} = \lambda_1 [N_1|_0] \, e^{-\lambda_1 t} \tag{27}$$

$$N_2 = \{- [N_1|_0]\} \, e^{-\lambda_1 t} + \{[N_2|_0] + [N_1|_0]\} \tag{28}$$

$$\frac{dN_3}{dt} = \lambda_2 [N_2|_0] \, e^{-\lambda_2 t} \tag{29}$$

$$N_3 = \left\{ -\frac{\lambda_2}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_1 t} + \left\{ - [N_2|_0] + \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_2 t} + \{[N_3|_0] + [N_2|_0] + [N_1|_0]\} \tag{30}$$

$$\frac{dN_4}{dt} = \lambda_3 [N_3|_0] \, e^{-\lambda_3 t} \tag{31}$$

$$\begin{aligned} N_4 = & \left\{ -\frac{\lambda_3}{\lambda_1} \frac{\lambda_2}{\lambda_3 - \lambda_1} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_1 t} + \left\{ -\frac{\lambda_3}{\lambda_3 - \lambda_2} [N_2|_0] + \frac{\lambda_3}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_2 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_2 t} \\ & + \left\{ - [N_3|_0] + \frac{\lambda_2}{\lambda_3 - \lambda_2} [N_2|_0] - \frac{\lambda_2}{\lambda_3 - \lambda_2} \frac{\lambda_1}{\lambda_3 - \lambda_1} [N_1|_0] \right\} e^{-\lambda_3 t} + \{[N_4|_0] + [N_3|_0] + [N_2|_0] + [N_1|_0]\} \end{aligned} \tag{32}$$

**Option Precise**

Generate the equations for $N_5$, $N_6$, and $N_7$ to determine the amount of ${}_{86}\mathrm{Rn}^{222}$ as a function of time.

**Option Sensible**

Consider ${}_{90}\mathrm{Th}^{234}$, and ${}_{91}\mathrm{Pa}^{234}$ transmutations as inconsequential and generate an equation for $N_5$.

# 1 "Creating" Radon

Most (> 99%) Uranium is ${}_{92}\mathrm{U}^{238}$, an unstable isotope that transmutes by spontaneous nuclear fission, and electron creation events generating the following transmutation sequence:

| Option Precise | Option Sensible | Process | Half-Life | "Decay" Constant |
|---|---|---|---|---|
| $N_1$ | $N_1$ | ${}_{92}\mathrm{U}^{238} \to {}_{2}\mathrm{He}^{4} + {}_{90}\mathrm{Th}^{234}$ | $\mathrm{t}_{1/2} = 4.86 \times 10^{9}$ year | $\lambda = 4.52 \times 10^{-18}$ s$^{-1}$ |
| $N_2$ | – Small – | ${}_{90}\mathrm{Th}^{234} \to {}_{-1}\mathrm{e}^{0} + {}_{91}\mathrm{Pa}^{234}$ | $\mathrm{t}_{1/2} = 24.1$ days | $\lambda = 3.33 \times 10^{-7}$ s$^{-1}$ |
| $N_3$ | – Small – | ${}_{91}\mathrm{Pa}^{234} \to {}_{-1}\mathrm{e}^{0} + {}_{92}\mathrm{U}^{234}$ | $\mathrm{t}_{1/2} = 6.75$ hours | $\lambda = 2.85 \times 10^{-5}$ s$^{-1}$ |
| $N_4$ | $N_2$ | ${}_{92}\mathrm{U}^{234} \to {}_{2}\mathrm{He}^{4} + {}_{90}\mathrm{Th}^{230}$ | $\mathrm{t}_{1/2} = 2.54 \times 10^{5}$ years | $\lambda = 8.65 \times 10^{-14}$ s$^{-1}$ |
| $N_5$ | $N_3$ | ${}_{90}\mathrm{Th}^{230} \to {}_{2}\mathrm{He}^{4} + {}_{88}\mathrm{Ra}^{226}$ | $\mathrm{t}_{1/2} = 8.0 \times 10^{4}$ years | $\lambda = 2.75 \times 10^{-13}$ s$^{-1}$ |
| $N_6$ | $N_4$ | ${}_{88}\mathrm{Ra}^{226} \to {}_{2}\mathrm{He}^{4} + {}_{86}\mathrm{Rn}^{222}$ | $\mathrm{t}_{1/2} = 1.60 \times 10^{3}$ years | $\lambda = 1.37 \times 10^{-11}$ s$^{-1}$ |
| $N_7$ | $N_5$ | ${}_{86}\mathrm{Rn}^{222} \to {}_{2}\mathrm{He}^{4} + {}_{84}\mathrm{Po}^{218}$ | $\mathrm{t}_{1/2} = 3.82$ days | $\lambda = 2.10 \times 10^{-6}$ s$^{-1}$ |

Radium is present in every natural geological location. The filters capture dust particles containing Radium. No isotopes of radium are stable and Ra-226 is the longest lived of those isotopes. Therefore, Radium acts as a reservoir for Radon-222, a noble gas, and its daughter isotopes that constitute a branching chain of transmutations that generate several $\alpha$, and $\beta$ particles.

$$< t >= \frac{\int_0^\infty dt \;\; te^{-\lambda t}}{\int_0^\infty dt \;\; e^{-\lambda t}} = \frac{\int_0^\infty d(\lambda t) \;\; (\lambda t)e^{-\lambda t}}{\lambda \int_0^\infty d(\lambda t) \;\; e^{-\lambda t}} = \frac{\int_0^\infty dx \;\; xe^{-x}}{\lambda \int_0^\infty dx \;\; e^{-x}} \tag{33}$$

$$\int_0^\infty dx \;\; xe^{-x} = -xe^{-x}\Big|_0^\infty + \int_0^\infty dx \;\; e^{-x} = \int_0^\infty dx \;\; e^{-x} \tag{34}$$

$$< t >= \lambda^{-1} \text{ , where , } \lambda = \frac{\ln 2}{t_{1/2}} \therefore t_{1/2} = \ln 2 < t > \approx 0.693 < t > \tag{35}$$

$$< t >= \tau \tag{36}$$